

The Chirality of Exceptional Points

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Exceptional points are singularities of the spectrum and wave functions which occur in connection with level repulsion. They are accessible in experiments using dissipative systems. It is shown that the wave function at an exceptional point is one specific superposition of two wave functions which are themselves specified by the exceptional point. The phase relation of this superposition brings about a chirality which should be detectable in an experiment.

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Level repulsion is a well known pattern in virtually all aspects of quantum mechanics. It states that the levels of a selfadjoint Hamiltonian H generically do not cross as a function of a parameter λ on which $H(\lambda)$ depends [1]. Its importance is particularly pronounced in the realm of quantum chaos [2,3]. The connection between exceptional points [4] and the occurrence of level repulsion has been discussed in [5].

An exceptional point (EP) is a value λ_c of the parameter λ , where two of the eigenvalues E_k of H are equal to each other – say $E_\nu(\lambda_c) = E_{\nu+1}(\lambda_c)$ – but where the space of the corresponding eigenvectors is only one-dimensional. We call this a coalescence of the eigenvalues and the eigenfunctions $|\psi_\nu\rangle, |\psi_{\nu+1}\rangle$. It is well known that this cannot occur for a selfadjoint Hamiltonian, where $E_\nu = E_{\nu+1}$ entails a two-dimensional space of eigenvectors, in which case the phenomenon is called a degeneracy.

Consider

$$H = H_0 + \lambda H_1, \quad (1)$$

where H_0, H_1 are real and symmetric $N \times N$ matrices, and let λ be a complex number. Then H is a complex symmetric matrix. At an EP there is always a singularity – namely a branch point – in the spectrum $E_k(\lambda)$ and the eigenfunctions $|\psi_k(\lambda)\rangle$. The spectrum consists of the values that one analytic function assumes on N Riemannian sheets. The sheets are connected by $N(N-1)$ square root branch points, the EP's. If an EP – connecting E_ν and $E_{\nu+1}$ – occurs sufficiently close to the real λ -axis, the two levels undergo a level repulsion as λ sweeps over the real axis in the vicinity of the EP. Conversely, when two levels undergo repulsion, there is always a nearby EP, where the expansions

$$E_\nu(\lambda) = E_\nu^0 + \sum_{s=1}^{\infty} e_s (\sqrt{\lambda - \lambda_c})^s$$

$$E_{\nu+1}(\lambda) = E_\nu^0 + \sum_{s=1}^{\infty} e_s (-\sqrt{\lambda - \lambda_c})^s \quad (2)$$

exist with a finite radius of convergence.

Three major results have been shown in [6] and experimentally verified in [7] when an EP is encircled in the complex λ -plane:

1. The two energy levels E_ν and $E_{\nu+1}$ connected at the EP are interchanged by a complete turn in the λ -plane.
2. The two wave functions $|\psi_\nu\rangle$ and $|\psi_{\nu+1}\rangle$ are not just interchanged like their eigenenergies but one of them undergoes a change of sign. In other words, a complete loop in the λ -plane leads to $\{\psi_\nu, \psi_{\nu+1}\} \rightarrow \{-\psi_{\nu+1}, \psi_\nu\}$. As an immediate consequence we conclude: (i) the EP is a fourth order branch point for the wave functions and (ii) different directions of going through the loop yield different phase behavior. In fact, encircling the EP a second time in the same direction we obtain $\{-\psi_\nu, -\psi_{\nu+1}\}$ while the next loop yields $\{\psi_{\nu+1}, -\psi_\nu\}$ and only the fourth loop restores the original pair $\{\psi_\nu, \psi_{\nu+1}\}$. It follows that the opposite direction yields after the first loop what is obtained after three loops in the former case.
3. The behavior of the two energy levels is distinctly different when a path in the λ -plane is taken below or above an EP. In one of the cases, the two levels avoid each other while their widths cross, in the other case, the two levels cross while their widths avoid each other.

In [7] the topological structure of an EP has been shown in the laboratory to be a physical reality. In the present paper, we focus attention upon the wave function at the EP. We show that the chiral behaviour that appears under item (2) above, is an intrinsic property of an EP. We argue that this chiral behaviour should be detectable in a suitable experiment.

Recall that for $\lambda \rightarrow \lambda_c$, one has $|\psi_\nu(\lambda)\rangle \rightarrow |\psi_{\text{EP}}\rangle$ and $|\psi_{\nu+1}(\lambda)\rangle \rightarrow |\psi_{\text{EP}}\rangle$ for the two coalescing wave functions. We mention that all the other $N-2$ wave functions are regular at a given EP.

Since H of Eq.(1) is not selfadjoint for complex λ , the right hand eigenvectors $|\psi_k\rangle$ are different from the left hand eigenvectors $\langle \tilde{\psi}_k |$. Both systems together form a biorthogonal basis, i.e. the completeness relation reads for $\lambda \neq \lambda_c$

$$\sum_k \frac{|\psi_k\rangle\langle\tilde{\psi}_k|}{\langle\tilde{\psi}_k|\psi_k\rangle} = 1. \quad (3)$$

Recall that

$$\langle\tilde{\psi}_j|\psi_{j'}\rangle = 0, \quad j \neq j'. \quad (4)$$

Due to the symmetric form of H the left hand eigenvector $\langle\tilde{\psi}|$ is just the complex conjugate of its right hand partner. Hence, in the Dirac notation, the (complex) components of the row vector $\langle\tilde{\psi}|$ coincide with the components of the column vector $|\psi\rangle$. From Eq.(4) it follows that

$$\langle\tilde{\psi}_{\text{EP}}|\psi_{\text{EP}}\rangle = 0 \quad (5)$$

since the orthogonality holds identically in λ and thus in particular at $\lambda = \lambda_c$, when $j = \nu$ and $j' = \nu + 1$. As a consequence, the inverse of the biorthogonal norm $\langle\tilde{\psi}_k|\psi_k\rangle$ that appears in Eq.(3), does not exist at $\lambda = \lambda_c$ for $k = \nu, \nu + 1$.

For a two-dimensional space, $N = 2$, one concludes from Eq.(5) that $|\psi_{\text{EP}}\rangle$ has the form

$$|\psi_{\text{EP}}\rangle \sim \begin{pmatrix} \pm i \\ 1 \end{pmatrix}. \quad (6)$$

Nowhere in the foregoing, a basis has been laid down with respect to which the coefficients of the vector are to be taken. In fact, (6) remains true under all orthogonal transformations of a given basis, even complex orthogonal ones. These are the transformations that conserve the symmetry of H which we consider. Hence, in every basis with respect to which H is symmetric, $|\psi_{\text{EP}}\rangle$ will have the form (6). In particular, there is no orthogonal transformation that maps $\begin{pmatrix} i \\ 1 \end{pmatrix}$ onto $\begin{pmatrix} -i \\ 1 \end{pmatrix}$. Every $|\psi_{\text{EP}}\rangle$ is therefore either $\sim \begin{pmatrix} i \\ 1 \end{pmatrix}$ or $\sim \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

For illustrative purpose we consider a two level model in detail. We stress, however, and below explicitly elaborate that even an infinite dimensional problem is, in the vicinity of an EP, locally equivalent to a two dimensional problem.

Consider

$$H = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} + \lambda U \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} U^T \quad (7)$$

with

$$U(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \quad (8)$$

where the angle ϕ and the energies ϵ_k, ω_k , $k = 1, 2$ are real. The eigenvalues are

$$E_{1,2}(\lambda) = \frac{\epsilon_1 + \epsilon_2 + \lambda(\omega_1 + \omega_2)}{2} \pm R, \quad (9)$$

where

$$R = \left\{ \left(\frac{\epsilon_1 - \epsilon_2}{2} \right)^2 + \left(\frac{\lambda(\omega_1 - \omega_2)}{2} \right)^2 + \frac{1}{2} \lambda(\epsilon_1 - \epsilon_2)(\omega_1 - \omega_2) \cos 2\phi \right\}^{1/2}. \quad (10)$$

The two levels coalesce when $R(\lambda)$ vanishes. This happens at

$$\lambda_c^\pm = -\frac{\epsilon_1 - \epsilon_2}{\omega_1 - \omega_2} \exp(\pm 2i\phi). \quad (11)$$

Note that for zero coupling ($\phi = 0$) the two branch points cancel each other and a genuine degeneracy occurs with the well known properties of a diabolic point [2].

The eigenfunctions can be parametrised by the complex angle θ as follows

$$|\psi_1(\lambda)\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad |\psi_2(\lambda)\rangle = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}. \quad (12)$$

Here, θ is related to the parameters in Eq.(7) via

$$\begin{aligned} \tan \theta(\lambda) &= \lambda(\omega_1 - \omega_2) \sin 2\phi / \\ &(E_1(\lambda) - E_2(\lambda) + \epsilon_1 - \epsilon_2 + \lambda(\omega_1 - \omega_2) \cos 2\phi). \end{aligned} \quad (13)$$

The eigenvectors of Eq.(12) are normalised in the biorthogonal sense

$$\langle\tilde{\psi}_k|\psi_k\rangle = 1 \quad \text{for } k = 1, 2, \lambda \neq \lambda_c. \quad (14)$$

An explicit calculation shows that the coefficients of the wave functions Eq.(12) diverge at the EP. Inserting λ_c^\pm into Eq.(13) we obtain

$$\tan \theta_c^\pm = \mp i.$$

This implies for $\lambda \rightarrow \lambda_c^\pm$

$$\begin{aligned} \cos \theta^\pm &\rightarrow \infty \\ \sin \theta^\pm &\rightarrow \mp i\infty. \end{aligned} \quad (15)$$

While the completeness relation Eq.(3) is obeyed for those values of λ which do not coincide with an EP, the set of eigenfunctions is incomplete at an EP. At the EP, the eigenfunctions (12) coalesce for $\lambda \rightarrow \lambda_c^\pm$ as

$$\begin{aligned} |\psi_1(\lambda)\rangle &\rightarrow F_1 \begin{pmatrix} \pm i \\ 1 \end{pmatrix}, \\ |\psi_2(\lambda)\rangle &\rightarrow F_2 \begin{pmatrix} \pm i \\ 1 \end{pmatrix}. \end{aligned} \quad (16)$$

Here, the factors $F_{1,2}$ depend on λ ; in fact they diverge for $\lambda \rightarrow \lambda_c$.

So far we have established that, in the two dimensional model, the wave function at the EP has, up to a complex factor, a strictly prescribed form: the ratio of the components is $+i$ at λ_c^+ and $-i$ at λ_c^- . This holds in

any basis and *irrespective of the parameters* ϵ_i , ω_i and ϕ . Before we turn to the physical relevance of this result we discuss a higher dimensional situation as this would usually prevail in an experimental set-up.

In higher dimensions the wave function at the EP will of course no longer have the simple form of Eq.(16), in fact $|\psi_{\text{EP}}\rangle$ has then N components. But using the completeness relation of Eq.(3) we expand

$$|\psi_{\text{EP}}\rangle = \sum_k c_k(\lambda) |\chi_k(\lambda)\rangle \quad (17)$$

with

$$c_k = \langle \tilde{\chi}_k | \psi_{\text{EP}} \rangle,$$

where

$$|\chi_k\rangle = \frac{|\psi_k\rangle}{\sqrt{\langle \tilde{\psi}_k | \psi_k \rangle}}$$

and

$$\langle \tilde{\chi}_k | = \frac{\langle \tilde{\psi}_k |}{\sqrt{\langle \tilde{\psi}_k | \psi_k \rangle}}$$

which ensures biorthogonal normalisation. If this expansion is used in close vicinity of an EP, where $|\psi_\nu\rangle$ and $|\psi_{\nu+1}\rangle$ are about to coalesce, it is obvious that only the terms in Eq.(17) with $k = \nu$ and $k = \nu+1$ make substantial contributions. In fact, all c_k vanish when $\lambda \rightarrow \lambda_c$ as follows from the orthogonality for $k \neq \nu, \nu+1$ and from Eq.(5) for $k = \nu, \nu+1$. However, the vanishing numerators for $k = \nu, \nu+1$ are compensated by the vanishing denominators with the result that in the limit $\lambda \rightarrow \lambda_c$ only the terms with $k = \nu$ and $k = \nu+1$ survive. This result implies that the N -dimensional vector $|\psi_{\text{EP}}\rangle$ is basically a superposition of only the two (N -dimensional) vectors $|\psi_\nu(\lambda)\rangle$ and $|\psi_{\nu+1}(\lambda)\rangle$; the closer λ is to λ_c the more correct is the statement. In other words, with regards to the EP, the N -dimensional problem can be locally simulated by a two-dimensional problem. From Eq.(16) we thus conclude that

$$\frac{c_\nu}{c_{\nu+1}} = +i \quad \text{or} \quad \frac{c_\nu}{c_{\nu+1}} = -i \quad (18)$$

must hold in the vicinity of λ_c but *independent* of λ within this vicinity.

A more explicit analytic consideration shows how this astounding and important result comes about. We denote the components of $|\psi_{\text{EP}}\rangle$ by $\{x_k\}$, $k = 1, \dots, N$ and recall (Eq.(5)) that $\sum_k x_k^2 = 0$. If λ is near to λ_c the components of the unnormalised $|\psi_\nu(\lambda)\rangle$ can be chosen as $\{x_k + d_k\}$ with $d_k = a_k \sqrt{\lambda - \lambda_c} + O(\lambda - \lambda_c)$, $k = 1, \dots, N$ and some constants a_k being of no interest here. The components of the unnormalised $|\tilde{\psi}_{\nu+1}(\lambda)\rangle$ must therefore, to lowest order in the d_k , have the form $\{x_k - d_k\}$. To lowest order in the d_k we obtain

$$\langle \tilde{\psi}_\nu | \psi_\nu \rangle = 2 \sum_k x_k d_k \quad (19)$$

$$\langle \tilde{\psi}_{\nu+1} | \psi_{\nu+1} \rangle = -2 \sum_k x_k d_k \quad (20)$$

$$\langle \tilde{\psi}_\nu | \psi_{\text{EP}} \rangle = \sum_k x_k d_k \quad (21)$$

$$\langle \tilde{\psi}_{\nu+1} | \psi_{\text{EP}} \rangle = - \sum_k x_k d_k \quad (22)$$

from which the statement of Eq.(18) immediately follows.

The local reduction – in the vicinity of an EP – of the full N -dimensional problem to an effective two dimensional problem is now achieved by the two-dimensional matrix $h = h_0 + \lambda h_1$ with the matrix elements

$$(h_0)_{jj'} = \langle \tilde{\chi}_j | H_0 | \chi_{j'} \rangle \quad (23)$$

$$(h_1)_{jj'} = \langle \tilde{\chi}_j | H_1 | \chi_{j'} \rangle, \quad j, j' = \nu, \nu+1 \quad (24)$$

using the relevant state vectors $|\chi_\nu\rangle$ and $|\chi_{\nu+1}\rangle$. In Fig.1 we display two different but typical examples to demonstrate how efficiently the procedure works. The eigenvalues of h_0 and h_1 yield the effective values of the ϵ_j and ω_j as used in Eq.(7). The effective coupling angle ϕ is obtained from the eigenvectors of h_1 in the basis where h_0 is diagonal (note that h_0 from Eq.(23) is not *a priori* diagonal). The straight lines in Fig.1 are the lines $\epsilon_j + \lambda \omega_j$ which correspond to the effective unperturbed lines ($\phi = 0$); switching on ϕ to the calculated value yields an almost exact approximation of the N -dimensional problem by the effective two-dimensional problem. For each level repulsion, i.e. for each EP, the procedure has to be carried out from the outset. The example of Fig.1 is based on a random ten dimensional case.

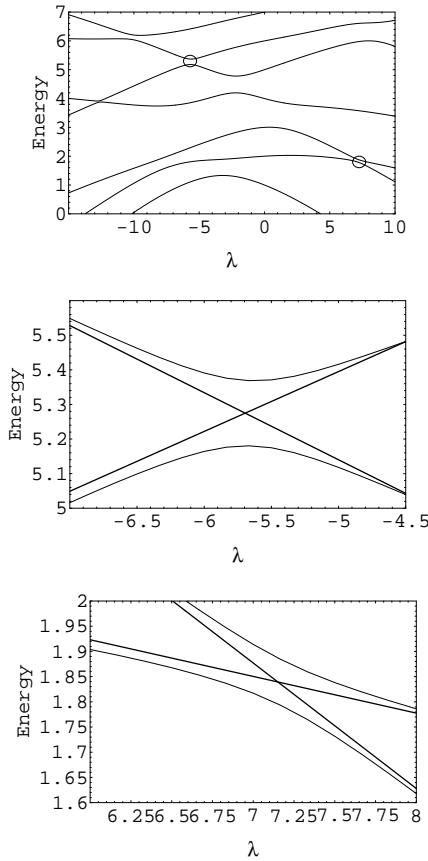


FIG. 1. Two-dimensional approximations of the EP associated with level repulsions. The top drawing displays a section of a ten-dimensional problem. The drawings in the middle and at the bottom are blow-ups of the encircled areas of the top. The straight lines are explained in the text. The distinction between the exact and the effective two-dimensional problem is within the line thickness for the curved lines in the middle and bottom drawing.

To summarise: an EP is locally equivalent to a two-dimensional problem. Knowing all parameters of the effective two-dimensional problem we know, from Eqs.(16) and (18), the specific superposition of the wave function at the EP in terms of those wave functions which coalesce at the EP. We find the relation

$$|\psi_{\text{EP}}\rangle = +i|\chi_\nu\rangle + |\chi_{\nu+1}\rangle \quad (25)$$

$$\text{for } \lambda_c^+ = -\frac{\epsilon_\nu - \epsilon_{\nu+1}}{\omega_\nu - \omega_{\nu+1}} e^{+2i\phi},$$

$$|\psi_{\text{EP}}\rangle = -i|\chi_\nu\rangle + |\chi_{\nu+1}\rangle \quad (26)$$

$$\text{for } \lambda_c^- = -\frac{\epsilon_\nu - \epsilon_{\nu+1}}{\omega_\nu - \omega_{\nu+1}} e^{-2i\phi}.$$

In a higher dimensional problem the quantities ϵ_j , ω_j and ϕ are effective quantities as defined above.

In an experimental situation like a microwave resonator, the phase factor $+i$ means that the time dependent wave function $|\chi_\nu\rangle$ has a leading phase of a quarter of a full period with respect to $|\chi_{\nu+1}\rangle$. For the phase $-i$ the wave is lagging by the same amount. This should

be detectable [8]. In the particular case, where the two wave functions can be associated with two independent linear polarisations, the wave function at the EP would then be an elliptic or circular wave with a definite chirality. We note that a similar observation has been made in [9] for the treatment of damped acoustic waves in a solid medium. If the two wave functions can be associated with different parities, the superposition again has a definite chirality.

We conclude that a definite chirality is associated with each EP. In a high dimensional problem one expects a random occurrence of a particular chiral behaviour just as the random occurrence of the associated level repulsions. Note that, in an experiment, only those EP are accessible which have a negative imaginary part of the eigenenergy. Depending on the effective values of the ϵ_j , ω_j and ϕ , these points may lie in the upper or lower λ -plane.

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